ON A PROBLEM OF G. KREWERAS

by

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ABSTRACT

In a simultaneous treatment of the problems of Young and of Simon Newcomb, Kreweras [6] obtained as a special case the numbers

$$c'_{r,s} \equiv \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} ,$$

for r,s positive integers, and pointed out many of their remarkable combinatorial properties. This thesis summarizes the main results of that paper, and gives two proofs of an expansion involving the $c'_{r,s}$ which, as Kreweras remarked, appears difficult to establish directly. The relationship of the numbers $c'_{r,s}$ to ballot problems, as first discussed by Narayana [10], [12], is reviewed in detail. Moreover, various results on lattice paths with diagonal steps are obtained by a uniform treatment suggested by well-known results on "ballot theorems".

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CHAPTER I

INTRODUCTION

The numbers

$$c'_{r,s} \equiv \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} \quad ,$$

for r,s positive integers, appear to have been introduced by Narayana [10]. In an expanded version of this paper [12], the relationship of the $c'_{r,s}$ to ballot problems and the numbers $\frac{1}{n+1}\binom{2n}{n}$, $n \geq 0$, which are sometimes called the "Catalan numbers", was fully discussed. We review this relationship in detail in the fourth chapter. Since it does not seem to be realized that many results on ballot theorems are directly applicable to other enumeration problems, we also derive in that chapter, by a uniform treatment based on Feller [2] and Narayana [10], [12], various results on lattice paths with diagonal steps. The formulae obtained by this unified method represent both a simplification and generalization of recently published formulae e.g. by Rohatgi [19] and Stocks [21]. Indeed, as will be evident from our approach, many results on lattice paths with diagonal steps can be derived using the well-known techniques for proving ballot theorems.

Considering the many and varied combinatorial interpretations of the so-called Catalan numbers, it is not surprising that the $c'_{r,s}$ - which represent a partition of the Catalan numbers - have arisen

independently on other occasions. Our second chapter consists of the main results of a paper by G. Kreweras [6]. He obtained the same numbers $c'_{r,s}$ as a special case in his elegant simultaneous treatment of the problems of Young and of Simon Newcomb. Furthermore, certain results in the theory of statistical estimation [13], [20] suggest that

$$(1.1) \qquad \sqrt{1-2u-2v-2uv+u^2+v^2} = 1-u-v-2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} c'_{r,s} u^r v^s \ .$$

As Kreweras remarked ([6] page 31), direct analytic proofs of (1.1) are of interest. Consequently, in the third chapter we present two such proofs of the above expansion, where, we recall once again,

$$c'_{r,s} \equiv \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1}$$

for r,s positive integers.

CHAPTER II

THE PROBLEMS OF YOUNG AND OF SIMON NEWCOMB

As was mentioned earlier, this chapter summarizes a paper by G. Kreweras [6] in which he obtained the numbers $c'_{r,s}$ as a special case. We emphasize that the main results are only stated, as the detailed proofs are sometimes too cumbersome.

2.1 THE PROBLEMS DEFINED.

The "problem of Simon Newcomb" is usually stated in the following manner: An individual is given a deck of cards of any "specification". That is to say, the deck may be composed in any manner whatever concerning the possible number of each of the cards whose face values may be any one of the integers $1,2,\ldots,h$. The cards are then dealt out in such a way as to start a new pile each time there appears a card of strictly less face value than the one which preceded it. The problem is to determine the number of distinct ways in which $r+1$ piles can be formed. Kreweras remarks that several different methods have been used to organize the calculation of the required number as a function of the non-negative integer r and of the sequence of positive integers $\alpha_1,\alpha_2,\ldots,\alpha_h$ which specify the composition of the deck. In this connection, he refers the reader to [18].

The "problem of Young" can be stated as follows: We are given two non-increasing finite sequences of non-negative integers (or sequences of Young) $Y = (y_1, y_2, \ldots, y_h)$ and $Y' = (y'_1, y'_2, \ldots, y'_h)$ of the same length h and such that Y "dominates" Y'; that is to say, such that each component of Y' is not greater than the corresponding component of Y. The problem is to determine the number of ways in which Y' can be joined to Y by a "Young chain". A Young chain is a succession of Young sequences such that each sequence can be obtained from the preceding one by adding a unit to only one term. Kreweras remarks that this problem has been resolved in the special case where all the terms of Y' are zero. The solution is achieved on one hand by an analytic formula due to Young [23] and on the other by an elegant algorithm introduced by Frame, Robinson and Thrall [3]. Kreweras [5] presents yet another solution of it in the general case of any Y' by expressing the required number $f(Y,Y')$ with the aid of a determinant of order h.

It is shown that the problems of Young and of Simon Newcomb can quite naturally be combined to form a unique problem which is more general. In effect, if one poses the problem of Young starting from Y and Y', each one of the chains of sequences of Young satisfying the conditions of the problem involve a certain number r (eventually zero) of "switchbacks"; a switchback being the case where, if U, V, W are three consecutive sequences of the Young chain $Y' \ldots UVW \ldots Y$, the term to increase in order to pass from V to W has strictly less index than the index of the term to increase in order to pass from U to V.

Example 2.1.1. Suppose $Y' = (210)$ and $Y = (322)$. Consider the following Young chain joining Y' to Y :

$$\begin{array}{ccccc} (210) & (220) & (221) & (321) & (322) \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ Y' & U & V & W & Y \end{array} .$$

In this chain, the index of the term to increase in order to pass from V to W is strictly less than the index of the term to increase in order to pass from U to V . Thus, the above chain has a switchback on the "occasion" of V . It is easy to verify, by examining every three consecutive sequences in the above chain, that no other switchbacks occur.

The total number $f(Y,Y')$ of Young chains joining Y' to Y can thus be decomposed into a sum

$$f(Y,Y') = \theta_o + \theta_1 + \ldots + \theta_r + \ldots \tag{2.1.1}$$

where θ_r counts those chains which involve r switchbacks. The general problem is to compute $\theta_r(Y,Y')$ knowing r, Y and Y' .

If we can solve the general problem, we can also solve the problem of Young (by summation of the θ_r with respect to r) and the problem of Simon Newcomb. For the specification $\alpha_1, \alpha_2, \ldots, \alpha_h$, the latter will simply be a particular case of the general problem. That is, it will suffice to define Y and Y' by:

$$
(2.1.2) \qquad
\begin{aligned}
y_1 &= \alpha_1 + \alpha_2 + \ldots + \alpha_h \\
y'_1 = y_2 &= \alpha_2 + \ldots + \alpha_h \\
&\ \vdots \\
y'_{h-1} = y_h &= \alpha_h \\
y'_h &= 0
\end{aligned}
\quad ,
$$

and then find the number of chains joining Y' to Y with r switch-backs. If conditions (2.1.2) are satisfied, every sequence of h integers dominating Y' term by term (in the non-strict sense) and dominated by Y will be a sequence of Young. The index of each term in a Young sequence will play the role of the face value of the card in the traditional formulation.

To obtain a solution to the general problem, Kreweras found it convenient to consider the sequences Y and Y' (which may if required be lengthened by adding some zero terms) as the elements of a distributive lattice $\mathbf{T}$ (lattice of Young cf [5]). This lattice $\mathbf{T}$ is that of infinite non-increasing sequences of non-negative integers which are of finite sum and in which the partial order denoted by $>$, though reflexive, is that of term by term domination (in the non-strict sense).

2.2 SEGMENTS AND POLYGONAL LINES IN THE LATTICE OF YOUNG.

Given two distinct elements R and S of $\mathbf{T}$ such that $S > R$ (or $R < S$), the segment RS is defined by Kreweras to be the set of elements X of $\mathbf{T}$ which satisfy the following conditions:

(i) $R < X < S$

(ii) there exists in the set N of natural integers an index p such that

$$i < p \Rightarrow x_i = s_i$$

$$i > p \Rightarrow x_i = r_i \quad .$$

The segment RS then possesses the following three properties which are easily verified:

(1) R and S belong to the segment RS (for R we have $p = 1$ and for S we can take p equal to the index above which all the r_i and s_i are zero and by consequence equal).

(2) If $\sum_{i \in N} r_i = \rho$ and $\sum_{i \in N} s_i = \sigma$, the segment RS is composed of $\sigma - \rho + 1$ distinct elements.

(3) The segment RS is totally ordered by the relation of order of the lattice T. If we enumerate in increasing order the elements of the segment, we pass from one to the following by adding one unit to the term which has the smallest possible subscript.

Example 2.2.1. If $R = (210)$ and $S = (322)$, we have $\rho = 3$ and $\sigma = 7$. The segment RS is composed in increasing order of the 5 elements (210) (310) (320) (321) (322).

The following property completes Kreweras' justification of the name "segment": If R' and S are distinct and if R' belongs to the segment RS and S belongs to the segment $R'S'$, then R' and S both belong to the segment RS'. To verify this, he proceeds

as follows: By (ii), if $R' \varepsilon$ segment RS then there exists a p such that

(2.2.1) $$i < p \Rightarrow r'_i = s_i$$

(2.2.2) $$i > p \Rightarrow r'_i = r_i$$

In the same way, if $S \varepsilon$ segment $R'S'$, then there exists a q such that

(2.2.3) $$i < q \Rightarrow s_i = s'_i$$

(2.2.4) $$i > q \Rightarrow s_i = r'_i \quad .$$

In addition we have $R' < S$, whence for all $i \varepsilon N$, $s_i \geq r'_i$; and since $S \neq R'$, there exists at least one index n for which $r'_n < s_n$. This index n can be neither less than p by (2.2.1) nor greater than q by (2.2.4). Hence $p \leq n \leq q$. It is now easy to show that $R' \varepsilon$ segment RS' (one would proceed in the same way for S). Condition (i) is satisfied since $R < R' < S < S'$. As for condition (ii) , it is satisfied with the integer p . In effect, $i < p \Rightarrow r'_i = s_i$ by (2.2.1) and since $p \leq q$ it is evident that $i < p \Rightarrow i < q \Rightarrow s_i = s'_i$ by (2.2.3); whence $i < p \Rightarrow r'_i = s'_i$. On the other hand $i > p \Rightarrow r'_i = r_i$ by (2.2.2).

The notion of segment being defined, three consecutive sequences $U\ V\ W$ are considered in a chain of Young joining Y' to Y such that in these sequences $v_i = u_i + 1$ and $w_j = v_j + 1$. It is pointed out that only two things can happen:

(1) If $j < i$ there is on the "occasion" (see Example 2.1.1) of V a switchback and V does not belong to the segment UW.

(2) If $j \geq i$ there is no switchback on the occasion of V, but on the other hand V belongs to the segment UW.

Hence the necessary and sufficient condition for a Young chain to allow r switchbacks on the occasions of $V_1V_2 \ldots V_r$ is that the sequences of Young of which it is composed generate the $r+1$ successive segments $Y'V_1, V_1V_2, \ldots, V_{r-1}V_r, V_rY$. The case where one of the segments forms with its predecessor a unique segment is excluded. Kreweras says, quite naturally, that these segments form a polygonal line with $r+1$ sides and $r+2$ vertices $Y', V_1, \ldots, V_r, Y$. That is, the required number $\theta_r(Y,Y')$ is also the number of ways of defining a polygonal line joining Y' to Y with exactly $r+1$ sides, or if one prefers, r vertices between Y' and Y.

2.3 RESOLUTION OF THE GENERAL PROBLEM.

Kreweras points out that under this last form, the problem is closely related to another problem given and resolved in [5]. Here we recall only the statement and final result: Given Y, Y' and r, how many r-chains are there in **T** joining Y' to Y. That is to say, how many sequences $Z_1Z_2 \ldots Z_r$ are there such that

$$(2.3.1) \qquad Y' < Z_1 < Z_2 < \ldots < Z_r < Y \; ?$$

He states that the answer to this question is an integer $W_r(Y,Y')$ equal to the determinant which has general element (row i, column j)

$$\binom{y_i - y'_j + r}{i - j + r}$$

For the justification, the reader is referred to [5].

If Y and Y' are given with $Y' < Y$, it is shown that there exists between the numbers $W_r(y,Y')$ which we know how to compute and the required number $\theta_r(Y,Y')$ a relation which can be stated precisely. In effect, in an r-chain such as that defined in (2.3.1), it can happen that for no index i the element Z_i belongs to the segment $Z_{i-1}Z_{i+1}$ (supposing $Z_o = Y'$ and $Z_{r+1} = Y$). In this case, such an r-chain immediately permits the definition of a polygonal line with $r+1$ sides joining Y' to Y. If this does not happen, the r-chain can be "thinned out" by taking out all the Z_i ($i = 1,2,\ldots,r$) which belong to a segment defined by two others among them. If in this way at most λ of the Z_i can be taken out, the $r-\lambda$ which remain define a polygonal line P with exactly $r-\lambda+1$ sides. The extreme case is where $\lambda = r$ and P is reduced to the segment $Y'Y$.

Conversely, if P is a polygonal line joining Y' to Y with $r-\lambda+1$ sides, there can be defined, beginning from P, an r-chain to which belong as particular terms the vertices of P. To do this, Kreweras remarks, it is necessary and sufficient to arbitrarily introduce λ new elements of $\mathbf{T}$ which may be distinct or not, vertices of P or not, but each of which belongs to a side of P. If $\sum_i y_i = \eta$ and $\sum_i y'_i = \eta'$, the union of the sides of P is obviously composed of $\eta-\eta'+1$ elements of $\mathbf{T}$. Since λ new elements can be placed

into $\eta-\eta'+1$ possible positions in $\binom{\eta-\eta'+\lambda}{\eta-\eta'}$ distinct ways, it is evident that for given Y and Y' one has:

$$(2.3.2) \qquad W_r = \theta_r + \binom{\eta-\eta'+1}{\eta-\eta'}\theta_{r-1} + \binom{\eta-\eta'+2}{\eta-\eta'}\theta_{r-2} + \ldots + \binom{\eta-\eta'+r}{\eta-\eta'}\theta_o \ .$$

These relations are easily solved for θ_r by observing that (2.3.2) leads to the following property of generating functions:

$$\sum_{r \geq o} W_r t^r = \frac{1}{(1-t)^{\eta-\eta'+1}} \sum_{r \geq o} \theta_r t^r \ .$$

Hence the general solution of the problem is

$$(2.3.3) \qquad \theta_r(Y,Y') = \sum_{k=o}^{r} (-1)^k \binom{\eta-\eta'+1}{k} W_{r-k}(Y,Y') \ ,$$

where the W_{r-k} are precalculated with the aid of the determinants mentioned above.

Example 2.3.1. If $Y' = (210)$ and $Y = (322)$, direct calculation by the method given in [5] yields $f(Y,Y') = 8$. The array below displays in the last column, with the aid of (2.3.3), the decomposition of these 8 chains according to formula (2.1.1)

r	W_r	θ_r
0	1	1
1	10	5
2	42	2

2.4 PARTICULAR CASES AND CONSEQUENCES.

Certain particular applications of the preceding results led Kreweras to make the interesting remarks which follow:

1. If one defines Y and Y' by (2.1.2) with all the α_i equal to 1, the problem becomes the "problem of Euler" which is the most simple particular case of the problem of Simon Newcomb. That is to say, how many arrangements of the $h!$ permutations of the integers $1,2,\ldots,h$ have r switchbacks? The θ_r of the answer are then the "numbers of Euler" of which many studies have been made (the reader is referred to [1] for an example). A table of these Euler numbers up to $h = 10$ is given in [18]. Now in this case the determinant which expresses $W_r(Y,Y')$ is easily computable since all the elements under the diagonal are zero. One obtains

$$W_r(Y,Y') = (r+1)^h \quad .$$

2. By taking all the α_i in (2.1.2) equal to the same natural integer α, the problem becomes that of Simon Newcomb "with the specification α^h". In the same way as before, the determinant which expresses W_r reduces to its diagonal product and one has

$$W_r = \binom{r+\alpha}{\alpha}^h \quad .$$

Also $\eta-\eta'$ is equal to αh and formulas (2.3.2) express some results established by Worpitzky [22].

Contrary to the general method of solving the problem of Simon Newcomb mentioned by Riordan [18], the method indicated above

does not require the possession (or the computation) of a table of Euler numbers.

3. If the two sequences $Y = (a,b)$ and $Y' = (a',b')$ of Young have nothing but non-zero terms, the calculation of $\theta_r(Y,Y')$ can be made without preliminary passage to the $W_r(Y,Y')$ owing to the fact that θ_r is given by the condensed expression

$$\theta_r = \binom{a-a'}{r}\binom{b-b'}{r} - \binom{b-a'+1}{r+1}\binom{a-b'-1}{r-1} . \tag{2.4.1}$$

This expression is quite easy to establish by induction on r. Setting $b = a = n+1$ and $a' = b' = 0$ in (2.4.1), one obtains

$$\begin{aligned} \theta_r &= \binom{n+1}{r}^2 - \binom{n+2}{r+1}\binom{n}{r-1} \\ &= \frac{n!(n+1)!}{r!(r+1)!(n-r)!(n-r+1)!} \\ &= \frac{(r+s)!(r+s+1)!}{r!(r+1)!s!(s+1)!} \qquad (s = n-r) \\ &= \frac{\binom{r+s+1}{r}\binom{r+s+1}{s}}{r+s+1} = c_{r,s} . \end{aligned} \tag{2.4.2}$$

The function (2.4.2) of r and s appears first to have been studied by Narayana [10]. A short table of these numbers is tabulated as follows:

	s=0	1	2	3	4	5	6	7
r=0	1	1	1	1	1	1	1	1
1	1	3	6	10	15	21	28	
2	1	6	20	50	105	196		
3	1	10	50	175	490			
4	1	15	105	490				
5	1	21	196					
6	1	28						
7	1							

It possesses a number of remarkable properties, notably that:

$$\sum_{r=0}^{n} c_{r,n-r} = \frac{(2n+2)!}{(n+1)!(n+2)!} \tag{2.4.3}$$

(this results almost immediately from the definition of $c_{r,s}$) and that under suitable conditions for convergence

$$\sum_{\substack{r \geq 0 \\ s \geq 0}} c_{r,s} u^r v^s = \frac{1-u-v-\sqrt{1-2(u+v)+(u-v)^2}}{2uv} \tag{2.4.4}$$

Kreweras remarks that

(1) although (2.4.4) is announced without complete proof in [13], it does not appear to be easy to establish by a purely analytic procedure.

(2) equation (2.3.2) gives a supplementary result owing to the fact that W_r (as is easy to establish in expanding the

corresponding determinant of order 2) is precisely equal to $c_{r,n}$.
From this observation, one obtains the remarkable identity

$$(2.4.5) \quad c_{r,n} = c_{r,n-r+1} + \binom{2n+1}{2n} c_{r-1,n-r} + \binom{2n+2}{2n} c_{r-2,n-r+1} + \cdots$$

$$+ \binom{2n+r}{2n} c_{o,n-1} \ .$$

This concludes the paper of Kreweras.

CHAPTER III

A DOUBLE SERIES EXPANSION

As we have seen in Chapter II, direct analytic proofs of the expansion

$$(3.A)\qquad \frac{1-u-v-\sqrt{1-2(u+v)+(u-v)^2}}{2uv} = \sum_{r=o}^{\infty} \sum_{s=o}^{\infty} c_{r,s}\, u^r v^s$$

where

$$c_{r,s} \equiv \frac{\binom{r+s+1}{r}\binom{r+s+1}{s}}{r+s+1}$$

are of interest. A little elementary algebra shows that expansion (3.A) is equivalent to the expansion

$$(3.B)\qquad \sqrt{1-2u-2v-2uv+u^2+v^2} = 1-u-v-2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} c'_{r,s}\, u^r v^s$$

where

$$(3.C)\qquad c'_{r,s} \equiv \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} .$$

In this chapter (see also [15]) we first present two proofs of the expansion in the form of (3.B) and then conclude with a few remarks. Before proceeding with the proofs however, we note that the numbers

generated by the function $c_{r,s}$ are identical to those generated by the function $c'_{r,s}$, except for a trivial modification in the initial values of r,s.

3.1 PROOF OF THE EXPANSION.

Our first proof was suggested to us by A. P. Guinand, and consists of verifying, by essentially a simple though tedious bit of algebra, that the series on the right hand side of (3.B) satisfies a certain differential equation. Consider

$$f(u,v) = \frac{1}{2}\{1-u-v-\sqrt{1-2u-2v-2uv+u^2+v^2}\} .$$

Clearly f satisfies the first order differential equation

$$(3.1.1) \qquad (1-2u-2v-2uv+u^2+v^2)\frac{\partial f}{\partial u} + (1-u+v)f = v+uv-v^2$$

with the boundary condition $f(o,v) = 0$ for all v. It is required to show that

$$(3.1.2) \qquad f(u,v) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} u^r v^s$$

for sufficiently small u,v. As this series obviously satisfies the boundary condition $f(o,v) = 0$ for all v, we need only show that it satisfies the differential equation (3.1.1). Noting from (3.1.2) that

$$f = uv+u^2v+uv^2 + \text{terms of higher order} ,$$

and

$$\frac{\partial f}{\partial u} = v+2uv+v^2 + \text{terms of higher order}$$

we have by substitution of these values into (3.1.1) and simplification (as far as second order terms in u,v),

$$(3.1.3) \qquad (1-2u-2v-2uv+v^2+u^2)\frac{\partial f}{\partial u}+(1-u+v)f = v+uv-v^2 + \text{higher order terms} .$$

Comparing (3.1.1) and (3.1.3) it remains to show that all terms of third and higher order on the right hand side of (3.1.3) vanish. We shall assume (or as can easily be verified in our second proof) that the series

(3.1.2)

$$f(u,v) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} u^r v^s$$

converges absolutely in a small neighborhood around the origin. Term by term differentiation of (3.1.2) with respect to u yields

$$(3.1.4) \qquad \frac{\partial f}{\partial u} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\binom{r+s-1}{r}\binom{r+s-1}{s}}{r+s-1} r\, u^{r-1} v^s \quad .$$

Consequently the coefficient of $u^r v^s$ in (3.1.3) for $r+s \geq 3$ is

$$(3.1.5) \qquad \begin{aligned} &(r+1)c'_{r+1,s} - 2rc'_{r,s} - 2(r+1)c'_{r+1,s-1} + (r-1)c'_{r-2,s} \\ &- 2rc'_{r,s-1} + (r+1)c'_{r+1,s-2} + c'_{r,s} - c'_{r-1,s} + c'_{r,s-1} . \end{aligned}$$

Substituting for $c'_{r,s}$ from (3.C) and simplifying by taking out a factor $\dfrac{[(r+s-2)!]^2}{[(r-1)!]^2[(s-1)!]^2}$ it is a simple bit of algebra to show that (3.1.5)

is equal to zero. From the remark after (3.1.3), we have completed the proof.

In our second proof we expand the left hand side of (3.B) and then proceed to show that this expansion is identical to the right hand side. We write the left hand side of (3.B) as

$$\begin{aligned} T &= \sqrt{(1-u-v)^2 - 4uv} \\ &= (1-u-v)y \\ &= y - (u+v)y \end{aligned}$$

where

$$y = \sqrt{1 - \frac{4uv}{(1-u-v)^2}} \quad . \tag{3.1.6}$$

Now for $-1 < \dfrac{4uv}{(1-u-v)^2} < 1$ we have

$$\begin{aligned} y &= \sum_{r=o}^{\infty} \binom{\frac{1}{2}}{r} \left(\frac{-4uv}{(1-u-v)^2} \right)^r \\ &= 1 - 2 \sum_{r=1}^{\infty} \frac{(2r-2)!}{r!(r-1)!} \frac{u^r v^r}{(1-u-v)^{2r}} \end{aligned} \tag{3.1.7}$$

where, of course, the series on the right hand side of (3.1.7) converges absolutely to y for u,v satisfying the specified condition (see, for example, [16], page 110). Again for $-1 < u+v < 1$ we find that

$$\begin{aligned} [1 - (u+v)]^{-2r} &= \sum_{s=o}^{\infty} \binom{-2r}{s} (-(u+v))^s \\ &= \sum_{s=o}^{\infty} \frac{(2r+s-1)!}{(2r-1)!s!} (u+v)^s \quad . \end{aligned} \tag{3.1.8}$$

Substituting (3.1.8) into (3.1.7) we have

$$y = 1 - 2 \sum_{r=1}^{\infty} \sum_{s=o}^{\infty} \frac{u^r v^r (2r+s-1)!(u+v)^s}{r!(r-1)!s!(2r-1)} \quad . \tag{3.1.9}$$

A little thought convinces us that the right hand side of (3.1.9) converges absolutely to y in the region of absolute convergence common to both (3.1.7) and (3.1.8). Hence

$$T = y - y(u+v)$$

$$= 1-u-v-2 \sum_{r=1}^{\infty} \sum_{s=o}^{\infty} \frac{u^r v^r (2r+s-2)!(u+v)^s}{r!(r-1)!s!} \tag{3.1.10}$$

$$= 1-u-v-2 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} u^r v^r \binom{2r+s-3}{r-1}\binom{r+s-2}{r-1} \frac{1}{r} (u+v)^{s-1} \quad .$$

We now show that the coefficient of $u^R v^S$ in (3.1.10) for $R \geq S$ is

$$\frac{\binom{R+S-1}{R}\binom{R+S-1}{S}}{R+S-1} \quad .$$

We note that, by considerations of symmetry, the requirement $R \geq S$ is no real restriction. Terms involving $u^R v^S$ will be obtained only when $r \leq S$, for if $r > S$ the powers of v are immediately too high. Now when $r = S$ and $s-1 = R-S$ the term involving $u^R v^S$ is given by

$$u^S v^S \binom{R+S-2}{S-1}\binom{R-1}{S-1} \frac{1}{S} \binom{R-S}{0} u^{R-S} v^o \quad .$$

Similarly when $r = S-1$ and $s-1 = R-S+2$, the term involving $u^R v^S$ is given by

$$u^{S-1}v^{S-1}\binom{R+S-2}{S-2}\binom{R}{S-2}\frac{1}{S-1}\binom{R-S+2}{1}u^{R-S+1}v \quad .$$

Continuing this procedure, we find that when $r = 1$ and $s-1 = R+S-2$ the term involving $u^R v^S$ is given by

$$uv\binom{R+S-2}{0}\binom{R+S-2}{0}\frac{1}{1}\binom{R+S-2}{S-1}u^{R-1}v^{S-1} \quad .$$

Thus, upon collecting terms, we can write the coefficient of $u^R v^S$ as

$$\sum_{k=1}^{S}\binom{R+S-2}{S-k}\binom{R+k-2}{S-k}\binom{R-S+2k-2}{k-1}\frac{1}{S-k+1}$$

(3.1.11)

$$= \frac{1}{R}\binom{R+S-2}{S-1}\sum_{k=1}^{S}\binom{S-1}{k-1}\binom{R}{S-k+1} \quad .$$

It remains to show that

$$\frac{1}{R}\binom{R+S-2}{S-1}\sum_{k=1}^{S}\binom{S-1}{k-1}\binom{R}{S-k+1} = \frac{\binom{R+S-1}{R}\binom{R+S-1}{S}}{R+S-1} \tag{3.1.12}$$

or equivalently, upon dividing both sides of (3.1.12) by $\frac{1}{R}\binom{R+S-2}{S-1}$ and making the substitution $K = k-1$ that

$$\sum_{K=o}^{S-1}\binom{S-1}{K}\binom{R}{S-K} = \binom{R+S-1}{S} \quad . \tag{3.1.13}$$

But (3.1.13) is a well-known identity; one proof is to note $(1+x)^R(1+x)^{S-1} = (1+x)^{R+S-1}$, and collect the coefficient of x^S on both sides. Hence our proof is complete.

3.2 CONCLUDING REMARKS.

Although it is easy enough to show the validity of the expansion (3.B) in a small neighborhood of the origin, we do not claim to have a sure method of making such double series expansions in general; nor have we investigated in detail the region of absolute convergence of the series. As Kreweras points out, such expansions are tedious (if not difficult) to establish directly. Indeed the expansion (3.B) was first suggested by simple considerations in the theory of estimation [20]. Since the connections between series expansions and UMV estimates have been adequately discussed, the interested reader can consult the references in [13].

CHAPTER IV

SOME INTERPRETATIONS AND GENERALIZATIONS OF THE NUMBERS $c'_{r,s}$

The "classical" ballot problem, that is to say the André-Poincaré "problème du scrutin" [17], can be stated as follows: In an election between two candidates A polls m votes, B polls n, m > n. If the votes are counted one by one, what is the probability that A leads B throughout the counting? As we mentioned earlier, this chapter is intended to review in detail the relationship of the $c'_{r,s}$ to the classical ballot problem and also to show how many results on ballot problems are directly applicable to other enumeration problems.

After some preliminary definitions and notations in section 4.1, a combinatorial problem is reviewed in section 4.2 in which Narayana [10] generalized the classical ballot problem using a partial order defined on the compositions of an integer. The relationship of the numbers $c'_{r,s}$ to the classical ballot problem is also indicated. In section 4.3 we generalize the classical ballot problem by giving essentially two different proofs of a "ballot problem with ties". Applications of the methods are then made to certain enumeration problems for lattice paths with diagonal steps in the plane. Without appealing de nouveau to the reflection principle or to specialized results on

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convolutions we show in section 4.4 that many of these results can be extended directly to three or more dimensions. In section 4.5 we conclude with a statement of the "duality principle" (Feller [2], Narayana [9]) for lattice paths with diagonal steps, and briefly indicate how many classical results could be generalized in a similar fashion.

4.1 DEFINITIONS AND NOTATIONS.

We give below a few definitions and notations due to Narayana [10].

Composition of an integer:

Given an integer n, an r-composition of n $(t_1, t_2, \ldots, t_r)$ is a set of t_i where $t_i \geq 1$ is an integer for $i = 1, 2, \ldots, r$ such that

$$t_1 + t_2 + \ldots + t_r = n.$$

In general, $(t_1, t_2, \ldots, t_r)$ and $(t_2, t_1, \ldots, t_r)$, where $t_1 + t_2 + \ldots + t_r = n$, are regarded as distinct r-compositions of n, unless $t_1 = t_2$. If r is an integer such that $1 \leq r \leq n$, there are obviously $\binom{n-1}{r-1}$ distinct r-compositions of n.

Relation of domination:

We say that an r-composition $(t_1, t_2, \ldots, t_r)$ of n dominates another r-composition $(t_1', t_2', \ldots, t_r')$ of n if and only if

$$
\begin{aligned}
t_1 &\geq t'_1 \\
t_1 + t_2 &\geq t'_1 + t'_2 \\
&\vdots \\
t_1 + \dots + t_{r-1} &\geq t'_1 + \dots + t'_{r-1} .
\end{aligned}
\tag{4.1.1}
$$

Evidently $t_1 + \dots + t_r = t'_1 + \dots + t'_r = n$.

Since the relation of domination defined by (4.1.1) is reflexive, transitive and antisymmetric, it represents a partial ordering of the r-compositions of n.

More generally, if $(t_1, t_2, \dots, t_r)$ is an r-composition of m and $(t'_1, \dots, t'_r)$ is an r-composition of n, where $m > n$, we say that $(t_1, \dots, t_r)$ dominates $(t'_1, \dots, t'_r)$ if the relations (4.1.1) are satisfied.

Numbering the $\binom{n-1}{r-1}$ r-compositions of n in some arbitrary order using the symbols $p_1, p_2, \dots, p_{\binom{n-1}{r-1}}$, let x_i be the number of compositions dominated by p_i in the set $p_1, p_2, \dots, p_{\binom{n-1}{r-1}}$; $i = 1, \dots, \binom{n-1}{r-1}$. Clearly the total (n,r) does not depend on the particular ordering chosen for numbering the r-compositions of n. We now state a lemma which was demonstrated by Narayana [10], [12] as a special case of a theorem which is motivated in the next section.

<u>Lemma 4.1.1</u> For all integers n,r such that $1 \leq r \leq n$,

$$(n,r) = \frac{\binom{n}{r}\binom{n}{r-1}}{n} = c'_{r,n-r+1} .$$

Before proceeding to the next section however, we give one more definition which will facilitate discussion of lattice paths in the plane.

Turn:

A turn occurs in a planar lattice path when a horizontal step or steps is followed by a vertical step or steps.

If one or more diagonal steps occur between a horizontal step and a vertical step we will again say that a turn has occurred. In other words, diagonal steps will be ignored in determining the number of turns in a path.

We note that a vertical step followed by one or more horizontal steps does not constitute a turn.

4.2 <u>RELATIONSHIP OF THE</u> $c'_{r,s}$ <u>TO THE CLASSICAL BALLOT PROBLEM.</u>

Following Narayana [10], [12], suppose we have a particle at the origin of an Euclidean space of k dimensions (k finite) and consider k mutually perpendicular axes. We shall suppose that the particle can move on the network of points p $(i_1, i_2, \ldots, i_k)$, where $i_1 \geq i_2 \geq \ldots \geq i_k \geq 1$ are all integers, according to the following rules:

Let $a_{i\alpha}$ be the increase in the i-th coordinate at the α-th step, such that,

(1) $a_{i\alpha} \geq 1$ for all i and α

(2) $a_{11} \geq a_{21} \geq \ldots \geq a_{k1} \geq 1$;

$a_{11} + a_{12} \geq a_{21} + a_{22} \geq \ldots \geq a_{k1} + a_{k2} \geq 2$

and in general for the α-th step

$$\sum_{j=1}^{\alpha} a_{ij} \geq \sum_{j=1}^{\alpha} a_{i'j} \quad (\geq\alpha) \quad \text{when} \quad i \leq i' \quad i,i' = 1,\ldots,k .$$

Let us suppose we know that at the r-th step the particle has reached the point $(a_1, a_2, \ldots, a_k)$, $a_1 \geq a_2 \geq \ldots \geq a_k \geq r$, and let $(a_1, \ldots, a_k)_r$ be the total number of possible paths by which the particle can arrive at this point. We have

<u>Theorem 4.2.1.</u>

$$(a_1, \ldots, a_k)_r = \begin{vmatrix} \binom{a_1-1}{r-1} & \binom{a_2-1}{r} & \cdots & \binom{a_k-1}{r+k-2} \\ \binom{a_1-1}{r-2} & \binom{a_2-1}{r-1} & \cdots & \binom{a_k-1}{r+k-3} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{a_1-1}{r-k} & \binom{a_2-1}{r-k+1} & \cdots & \binom{a_k-1}{r-1} \end{vmatrix} .$$

We prove Theorem 4.2.1 as first given in [10] for $k = 2$ and then show that Lemma 4.1.1 is a special case. The general proof of Theorem 4.2.1 is given as a special case in [11]. For $k = 2$ however, the network consists of the points $p(i,j)$ where $i \geq j \geq 1$.

For $r = 1$, obviously $(a_1,a_2)_1 = 1$ for $a_1 \geq a_2 \geq 1$

$= 0$ otherwise.

For $r = 2$,

$$
\begin{aligned}
(a_1,a_2)_2 &= 0 \quad \text{if} \quad a_1 < a_2 \quad \text{or} \quad a_2 < 2 \\
&= \sum_{i=1}^{a_1-1} \sum_{j=1}^{a_2-1} (i,j)_1 \quad \text{if} \quad a_1 \geq a_2,\ a_2 \geq 2 \\
&= \sum_{i \geq j}^{a_1-1} \sum_{j=1}^{a_2-1} 1 \\
&= \sum_{j=1}^{a_2-1} (a_1-j) \\
&= (a_1-1)(a_2-1) - \binom{a_2-1}{2} .
\end{aligned}
$$

Thus for $r = 1,2$, the theorem is true. So, applying induction, we get

$$
\begin{aligned}
(a_1,a_2)_r &= \sum_{i=1}^{a_1-1} \sum_{j=1}^{a_2-1} (i,j)_{r-1} = \sum_{i \geq j}^{a_1-1} \sum_{j=1}^{a_2-1} \left[\binom{i-1}{r-2}\binom{j-1}{r-2} - \binom{i-1}{r-3}\binom{j-1}{r-1} \right] \\
&= \sum_{j=1}^{a_2-1} \left[\binom{j-1}{r-2}\binom{j-1}{r-2} - \binom{j-1}{r-3}\binom{j-1}{r-1} + \binom{j}{r-2}\binom{j-1}{r-2} - \binom{j}{r-3}\binom{j-1}{r-1} + \dots \right. \\
&\qquad \left. + \binom{a_1-2}{r-2}\binom{j-1}{r-2} - \binom{a_1-2}{r-3}\binom{j-1}{r-1} \right]
\end{aligned}
$$

(4.2.1)

$$
\begin{aligned}
&= \sum_{j=r-1}^{a_2-1} \binom{j-1}{r-2} \left[\binom{j-1}{r-2} + \binom{j}{r-2} + \dots + \binom{a_1-2}{r-2} \right] - \sum_{j=r}^{a_2-1} \binom{j-1}{r-1} \left[\binom{j-1}{r-3} \right. \\
&\qquad \left. + \binom{j}{r-3} + \dots + \binom{a_1-2}{r-3} \right] \\
&= \sum_{k=o}^{a_2-r} \binom{k+r-2}{r-2} \left[\binom{k+r-2}{r-2} + \binom{k+r-1}{r-2} + \dots + \binom{a_1-2}{r-2} \right] \\
&\qquad - \sum_{k=o}^{a_2-r-1} \binom{k+r-1}{r-1} \left[\binom{k+r-1}{r-3} + \dots + \binom{a_1-2}{r-3} \right] .
\end{aligned}
$$

Using the well-known identity

$$\sum_{\nu=o}^{n} \binom{a-\nu}{r} = \binom{a+1}{r+1} - \binom{a-n}{r+1}$$

we find that (4.2.1) becomes

$$\begin{aligned}(a_1,a_2)_r = &\left[\binom{a_1+1}{r-1} - \binom{a_1}{r-2} - \binom{a_1-1}{r-2}\right]\left[\binom{a_2+1}{r-1} - \binom{a_2-1}{r-2} - \binom{a_2}{r-2}\right] \\ &- \left[\binom{a_2+1}{r} - \binom{a_2-1}{r-1} - \binom{a_2}{r-1}\right]\left[\binom{a_1+1}{r-2} - \binom{a_1}{r-3} - \binom{a_1-1}{r-3}\right] .\end{aligned} \tag{4.2.2}$$

The identity

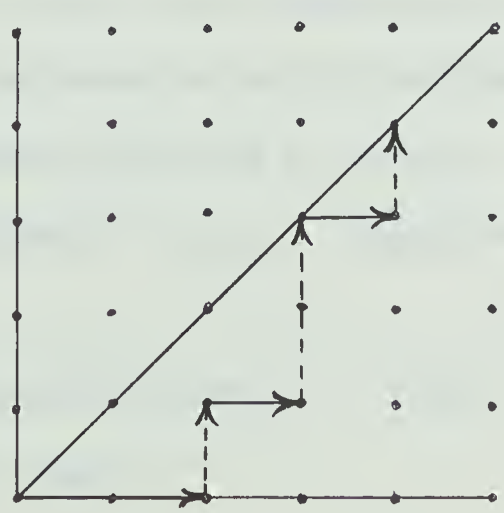
$$\binom{x+1}{r} = \binom{x}{r-1} + \binom{x}{r}$$

applied to (4.2.2) gives

$$(a_1,a_2)_r = \binom{a_1-1}{r-1}\binom{a_2-1}{r-1} - \binom{a_1-1}{r-2}\binom{a_2-1}{r}$$

as required.

As Narayana first showed [10], in setting $a_1 = a_2 = n$, each path represents a domination of an r-composition of n by an r-composition of n. We give below an example for $n = 4$, $r = 3$. (2,1,1) dominates (1,2,1).



For $a_1 = a_2 = n \geq r$, the result of Lemma 4.1.1 is obtained since in this case

$$(n,r) = (a_1, a_2)_r$$

or

$$(n,r) = \frac{\binom{n}{r}\binom{n}{r-1}}{n}, \quad (1 \leq r \leq n)$$

$$= c'_{r,n-r+1} .$$

The proof of Lemma 4.1.1 makes it clear that the numbers $c'_{r,n-r+1}$ can be interpreted as the number of paths from $(0,0)$ to (n,n) not crossing the line $y = x$ such that each path has exactly r turns.

The remarkable property (2.4.3) mentioned by Kreweras now appears in a very natural setting. Furthermore, since

$$\sum_{r=1}^{n} c'_{r,n-r+1} = \frac{1}{n+1}\binom{2n}{n},$$

the numbers $c'_{r,n-r+1}$ can rightfully be called a "partition" of the Catalan numbers.

An alternative well-known interpretation of the numbers $c'_{r,s}$ is as follows: If in the classical ballot problem we represent a vote for A by a unit horizontal step and a vote for B by a unit vertical step, we see that the numbers $c'_{r,n-r+1}$ represent the number of voting patterns in which

(1) each candidate receives n votes,

(2) B never leads A

and (3) during the count, there are r occasions on which a vote or votes for A is followed by a vote or votes for B.

4.3 THE BALLOT PROBLEM WITH TIES.

Consider the following ballot problem, which for $r = 0$ reduces to the classical "problème du scrutin": Let candidate A poll $(m - r)$ votes, candidate B poll $(n - r)$ votes and let a further r votes be counted for both A and B. These r votes represent ties, and so the total number of votes for A, B are m, n respectively, where of course $m > n$ and $0 \leq r \leq n$. What is the probability that A leads B throughout the counting?

We give two proofs that the required probability is $\frac{m-n}{m+n-r}$ if $0 \leq r \leq n$. Our first proof follows almost verbatim the classical proof of Poincaré [17], Feller [2], while the second utilizes the simple idea of "placing balls into cells" used in occupancy problems. It will be seen in sections 4.4 and 4.5 that the methods used in our second proof are particularly useful in solving certain lattice path enumeration problems in three or more dimensions and in generalizing many classical results involving enumeration of lattice paths. Following the second proof we illustrate more explicitly the ways in which A can hold a lead over B throughout the count by repeating some arguments in [12]. A few remarks and a corollary conclude the section.

We now state formally as Theorem 4.3.1 the ballot problem with ties.

Theorem 4.3.1. The probability that A leads B throughout the counting when A has m votes, B has n votes, $m > n$ and $0 \leq r \leq n$ votes are ties is $\frac{m-n}{m+n-r}$.

Proof 1. We prove the theorem for $0 \leq r < n$, noting that the case $r = n$ is trivial and can be easily established. Let a vote for A be represented by a unit horizontal step, a vote for B by a unit vertical step and a vote for both A and B by a diagonal step. Surely the number of distinct paths from $(0,0)$ to (m,n) with $(m-r)$ horizontal steps, $(n-r)$ vertical steps and r diagonal steps is $(m,n,r) = \frac{(m+n-r)!}{(m-r)!(n-r)!r!}$. Define four mutually exclusive and exhaustive types of paths from $(0,0)$ to (m,n) as follows:

(1) An α path starts with a unit vertical step.

(2) A β path starts with a unit horizontal step and then at some other step touches or crosses the line $y = x$.

(3) A γ path starts with a diagonal step.

(4) A δ path starts with a horizontal step and remains below the line $y = x$.

The number of α paths is equivalent to the number of paths from $(0,1)$ to (m,n) with $(n-r-1)$ vertical steps, $(m-r)$ horizontal steps and r diagonal steps which is $\frac{(m+n-r-1)!}{(m-r)!(n-r-1)!r!}$. Using the reflection principle which the probability literature attributes to D. André (1887) we see that the β paths are in one-to-one correspondence with the α paths and hence are the same in number. The number of γ paths is equivalent to the number of paths from $(1,1)$ to (m,n) with $(m-r)$ horizontal steps, $(n-r)$ vertical steps and

(r-1) diagonal steps which is $\frac{(m+n-r-1)!}{(m-r)!(n-r)!(r-1)!}$. Thus the number of δ paths is

$$(4.3.1) \qquad \frac{(m+n-r)!}{(m-r)!(n-r)!r!} - \frac{2(m+n-r-1)!}{(m-r)!(n-r-1)!r!} - \frac{(m+n-r-1)!}{(m-r)!(n-r)!(r-1)!} ;$$

or dividing (4.3.1) by the total number of paths (m,n,r) we see that the required probability is $\frac{m-n}{m+n-r}$. Our first proof is complete.

<u>Proof 2</u>. The number of paths to $(m-r,n-r)$ lying below the line $y = x$ and having only horizontal and vertical steps is by the classical ballot problem

$$(4.3.2) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r} .$$

Since each path to $(m-r,n-r)$ passes through $(m+n-2r+1)$ lattice points, which for convenience we refer to as cells, we see that there are $(m+n-2r)$ cells in which the r diagonal steps can be placed. The excluded cell is of course the origin, since starting with a diagonal step violates the conditions of the ballot problem. The number of ways of placing r diagonal steps into $(m+n-2r)$ cells is

$$(4.3.3) \qquad \binom{m+n-r-1}{r} .$$

Hence the total number of paths to (m,n) satisfying the conditions of the theorem is from (4.3.2) and (4.3.3)

$$(4.3.4) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n-r-1}{r} .$$

Division of (4.3.4) by the total number of paths (m,n,r) yields the result $\frac{m-n}{m+n-r}$ as before.

We can be more specific about the structure of the paths from $(0,0)$ to (m,n) using the idea of partial orders through the relation of domination. We prove the following

<u>Theorem 4.3.2.</u> The number of paths from $(0,0)$ to (m,n) lying below the line $y = x$ with r $(0 \leq r < n)$ diagonal steps and $k \geq 1$ turns is

$$\binom{m+n-r-1}{r}\left[\binom{n-r-1}{k-1}\binom{m-r-1}{k} - \binom{n-r-1}{k}\binom{m-r-1}{k-1}\right]. \tag{4.3.5}$$

<u>Proof.</u> We first determine the number of paths with k turns and no diagonal steps to the point $(m-r,n-r)$ such that each path lies entirely below the line $y = x$. We note that A can hold a lead over B in the following mutually exclusive ways:

(1) The last vote is for B.

(2) The last vote is for A; but the last but one is for B.

(3) The last two votes are for A; the preceding one is for B.

⋮

(m-n) The last $(m-n-1)$ votes are for A; the preceding one is for B.

We saw in section 4.2 that the number of paths to $(m-r,n-r)$ with k turns and not crossing the line $y = x$ such that the last vote is for B is

$$\binom{m-r-1}{k-1}\binom{n-r-1}{k-1} - \binom{m-r-1}{k-2}\binom{n-r-1}{k}.$$

But this is equivalent to the number of paths to $(m-r+1,n-r)$ lying entirely beneath the line $y = x$. Hence the number of paths to $(m-r,n-r)$ satisfying the required conditions, the last vote being for B is

$$\binom{m-r-2}{k-1}\binom{n-r-1}{k-1} - \binom{m-r-2}{k-2}\binom{n-r-1}{k} .$$

In the same way, the case where the last vote is for A and the last but one is for B gives rise to

$$\binom{m-r-3}{k-1}\binom{n-r-1}{k-1} - \binom{m-r-3}{k-2}\binom{n-r-1}{k} \quad \text{paths.}$$

If just the last $(m-n-1)$ votes are for A, we have

$$\binom{n-r-1}{k-1}\binom{n-r-1}{k-1} - \binom{n-r-1}{k-2}\binom{n-r-1}{k} \quad \text{paths}$$

satisfying the required conditions. Hence the total number of paths to $(m-r,n-r)$ with k turns such that each path lies entirely below the line $y = x$ is

$$(4.3.6) \qquad \binom{n-r-1}{k-1}\sum_{i=2}^{m-n+1}\binom{m-r-i}{k-1} - \binom{n-r-1}{k}\sum_{i=2}^{m-n+1}\binom{m-r-i}{k-2} .$$

Applying the well-known identity

$$\sum_{\nu=0}^{n}\binom{a-\nu}{r} = \binom{a+1}{r+1} - \binom{a-n}{r+1} ,$$

we see that (4.3.6) reduces to

$$(4.3.7) \qquad \binom{n-r-1}{k-1}\binom{m-r-1}{k} - \binom{n-r-1}{k}\binom{m-r-1}{k-1} .$$

Multiplying (4.3.7) by the number of ways of placing r balls into

$(m+n-2r)$ cells, we see that the required number of paths is given by (4.3.5).

Remarks.

1. Since it is a trivial matter to calculate the number of paths to (m,n) with $r = n$ diagonal steps, we have not stated this case here.

2. Summing (4.3.5) over k from 1 to $(n-r)$ yields, after some simplification, $\frac{(m+n-r-1)!(m-n)}{r!(m-r)!(n-r)!}$ paths to (m,n) with r diagonal steps, as in (4.3.4).

3. Clearly Theorem 4.3.2 represents a considerable generalization of the ballot problem (without using the reflection principle).

As a sample of other results which follow immediately from Theorem 4.3.2 we now state a corollary with an idea of the proof briefly indicated.

Corollary 4.3.1. The number of paths from $(0,0)$ to (n,n) lying entirely below the line $y = x$ (except of course for the end points) with (a) k turns $(k \geq 1)$, (b) r diagonal steps $(0 \leq r \leq n-2)$ is

$$(4.3.8) \qquad \binom{2n-r-2}{r}\left[\binom{n-r-2}{k-1}\binom{n-r-1}{k} - \binom{n-r-2}{k}\binom{n-r-1}{k-1}\right].$$

Observe that a path satisfying the above requirements must end with a vertical step so that we are essentially dealing with paths below the diagonal to $(n,n-1)$.

We remark that (4.3.8) clearly represents a generalization of Moser and Zayachkowski [8] page 225, equation 2.8. This can be seen

by summing (4.3.8) over all k and then over all r. Similarly, expressions can be obtained for $Q(m,n)$ and $Q'(m,n)$ defined by Rohatgi in [19] for all cases $m \geq n$.

4.4 LATTICE PATHS IN E^3 WITH CUBE DIAGONAL STEPS.

In this section we illustrate the simplicity with which the "balls into cells" technique can be used to solve lattice path problems of a more general nature than those discussed in the previous section. In three dimensions, a diagonal step may be taken as a cube diagonal. Letting m, n, k be non-negative integers, each lattice path has steps of the following types:

(1) x-increasing only, e.g. $[(m,n,k), (m+1,n,k)]$,

(2) y-increasing only, e.g. $[(m,n,k), (m,n+1,k)]$,

(3) z-increasing only, e.g. $[(m,n,k), (m,n,k+1)]$,

(4) cube diagonal, e.g. $[(m,n,k), (m+1,n+1,k+1)]$.

Without appealing to the reflection principle we now prove a theorem which generalizes the theorems in Stocks [21], and conclude this section with a few corollaries and remarks.

<u>Theorem 4.4.1.</u> The number of paths from $(0,0,0)$ to (m,n,k), $m > n$, with r cube diagonal steps such that each component of each path lies entirely to the $(m,0,0)$ side of the diagonal plane $y = x$ is

$$\frac{(m-n)(m+n+k-2r-1)!}{(m-r)!(n-r)!(k-r)!r!} \quad . \tag{4.4.1}$$

<u>Proof</u>. The number of paths to $(m-r,n-r,0)$ lying entirely to the $(m,0,0)$ side of the diagonal plane $y = x$ is by the classical ballot problem

$$(4.4.2) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r} .$$

Now $(k-r)$ vertical steps can be placed into $(m+n-2r)$ cells in

$$(4.4.3) \qquad \binom{m+n+k-3r-1}{k-r} \quad \text{ways} .$$

Hence by (4.4.2) and (4.4.3) the number of paths to $(m-r,n-r,k-r)$ without cube diagonal steps is

$$(4.4.4) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n+k-3r-1}{k-r} .$$

Placing r cube diagonal steps into the available $(m+n+k-3r)$ cells in

$$(4.4.5) \qquad \binom{m+n+k-2r-1}{r} \quad \text{ways,}$$

we see from (4.4.4) and (4.4.5) that the total number of paths to (m,n,k) satisfying the stated conditions is

$$(4.4.6) \qquad \frac{m-n}{m+n-2r}\binom{m+n-2r}{m-r}\binom{m+n+k-3r-1}{k-r}\binom{m+n+k-2r-1}{r} .$$

A simple computation reduces (4.4.6) to (4.4.1).

As a sample of other results which follow immediately from Theorem 4.4.1, we now state two corollaries with an idea of their proofs briefly indicated.

[illegible]

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[illegible]

[illegible]

Corollary 4.4.1. The number of paths from $(0,0,0)$ to (n,n,n) with r cube diagonal steps such that, excepting of course the end points, each component of each path lies entirely to the $(n,0,0)$ side of the diagonal plane $y = x$ is

$$(4.4.7) \qquad \frac{(3n-2r-2)!}{(n-r)^2[(n-r-1)!]^3 r!} \quad .$$

The proof is immediate if one observes that the required number of paths is equivalent to the number of paths to $(n,n-1,n)$ lying entirely to the $(n,0,0)$ side of the diagonal plane $y = x$.

Corollary 4.4.2. The number of paths from $(0,0,0)$ to (m,n,k), $m \geq n$, such that no path has a component on the non-$(m,0,0)$ side of the diagonal plane $y = x$ is

$$(4.4.8) \qquad \frac{(m-n+1)(m+n+k-2r)!}{(m-r+1)!(n-r)!(k-r)!r!} \quad .$$

To prove Corollary 4.4.2 observe that the required number of paths is equivalent to the number of paths to $(m+1,n,k)$ lying entirely to the $(m+1,0,0)$ side of the diagonal plane.

Remarks.

1. Summing (4.4.7) over r from 0 to $(n-1)$ yields a much simpler expression than that obtained by Stocks [21], page 656, for the number of paths to (n,n,n) with components to the $(n,0,0)$ side of the diagonal plane. A little elementary algebra however, simplifies Stocks' expression to ours.

2. Setting $m = n = k$ in (4.4.8) and summing over r from 0 to n clearly simplifies another expression obtained by Stocks [21],

[illegible]

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page 658, for the number of paths to (n,n,n) not crossing the diagonal plane. Again it is a simple matter to show equivalence of his expression to ours.

4.5 REMARKS ON FURTHER STUDY.

It is clear that by using Theorem 4.2.1, which is also valid in higher dimensions, we can obtain other results pertaining to enumeration of lattice paths. Since our formulation of the ballot problem with ties is consistent with the duality principle for ballot problems as in Narayana [9] and Feller [2], a result on first passages with ties can be obtained as follows: Let us assume that each step in a ballot problem with ties takes a unit of time. By duality we see immediately that the probability that a first passage through $(m-n)$ occurs at time $m+n-r$ is $\frac{m-n}{m+n-r}$ where $m > n \geq r$. We suspect that the same technique of placing balls into cells can be applied to other results in Chapter III of Feller [2] as well but it does not seem worthwhile to do so here. Of course the balls into cells technique provides an alternative expression for the main result in [8] using, for example, the results of Grossman [4].

Perhaps essentially new results can be obtained by trying to extend this technique to further refinements of the ballot problem, for example, 'An Analogue of the Multinomial Theorem' by Narayana [14]. This appears to have been partially studied by Mohanty and Handa [7], although in a different direction.

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